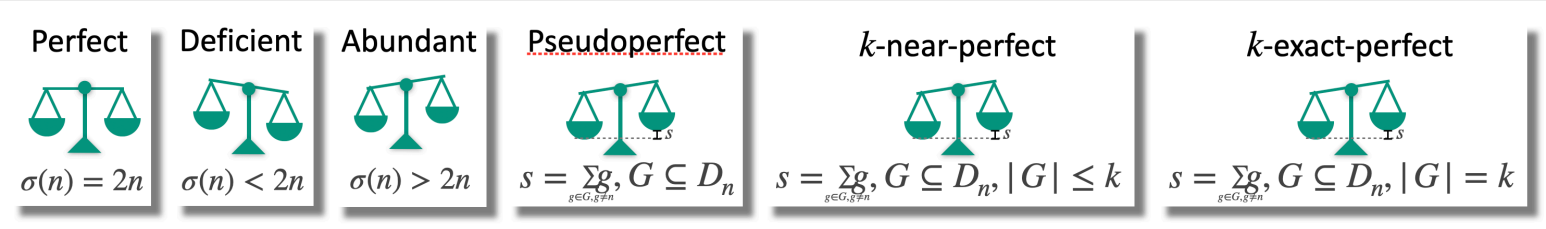
Redundancy Degree and Reconstructibility in k-Exact-Perfect Numbers

- Pranav Krishnapuram



✦ This number-theoretic research introduces a novel framework to classify a *k-exact-perfect number* (*k-epn*) by the structure of its *redundant divisors*. We define two properties to classify *k*-epns:
- *Redundancy Degree:* counts the number of redundant sets of *k redundant divisors*.
- *Reconstructibility:* determines whether the least-common-multiple of divisors of all size-*k* redundant sets equals *n*.

✦ We study these properties for 2-epns and 3-epns and show through computational search that, as redundancy degree increases,
- the density of *k*-epns decreases, and
- the likelihood of reconstructibility increases.

✦ Through both analysis and computational search in $n = 2^a p$ for odd prime $p < 2^{a+1}$, we show that,
- Among 2-epns, exactly one subfamily with Mersenne primes has redundancy degree greater than 1 and satisfies reconstructibility.
- Among 3-epns, multiple subfamilies have redundancy degrees up to 3 and reconstructibility.
- All sub-families can be fully expressed through formulas for their redundant divisors.

Introduction

Work by Others (Literature Survey)

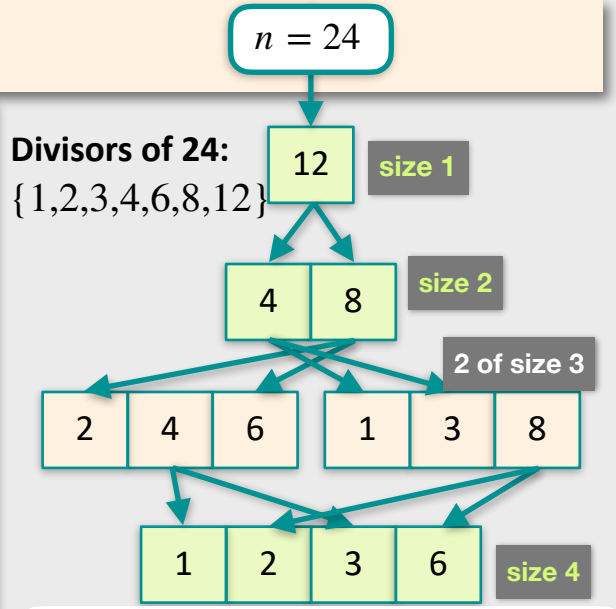
- [2],[3] defined a *k*-near-perfect-number (*k*-npn) as a number with at least k divisors that sum to the abundance of the number.
  - The k divisors are called "redundant divisors" and the set of the k redundant divisors are called "redundant set".
- [1],[2],[3] studied $n = 2^a p$, $n = 2^a pq$, $n = 2^a p^i$.[2],[3] derived solutions for asymptotic density of k-npns.
- [4] defined *k*-exact-perfect-numbers (*k*-epns) as integers with exactly k redundant divisors.

Project Origin

- I got inspiration for this project from [1], a publication by high schoolers on 2-near-perfect-numbers.
- Preliminary computational search revealed rich patterns beyond traditional classifications. I found multiple redundant sets of size 2, 3. So, I defined "redundancy degree" to denote the multiplicity of redundant sets.
- Multiple redundant sets expose more divisors. I wondered how information about the number is embedded in the divisors and if multiplicity affects how the divisors can reconstruct the number. So, I defined "reconstructibility".

Research Questions

- **Key questions** based on thought-provoking patterns in redundant sets:
  - Can n have multiple redundant sets of size k? For which n does multiplicity exist? How large is the multiplicity? How does the density of k-epns vary with multiplicity.
  - As the multiplicity increases, more divisors of n are revealed through the redundant sets. Is there enough information in the redundant sets to reconstruct n? Which n are reconstructible? How to reconstruct n?
- **Research Focus:** $k = 2,3$ and simple-to-analyze numbers of the form $n = 2^a p$
- **Hypotheses:**
  - Asymptotic density of numbers with a multiplicity of redundant sets is 0.
  - Numbers with higher multiplicity will be rarer than those will lower multiplicity.
  - Numbers with higher multiplicity will have denser reconstructibility.



Redundant Sets for $n = 24$: One each of sizes 1, 2, 4, two of size 3. Some sets jointly reveal n.

Framework: Background and Terminology

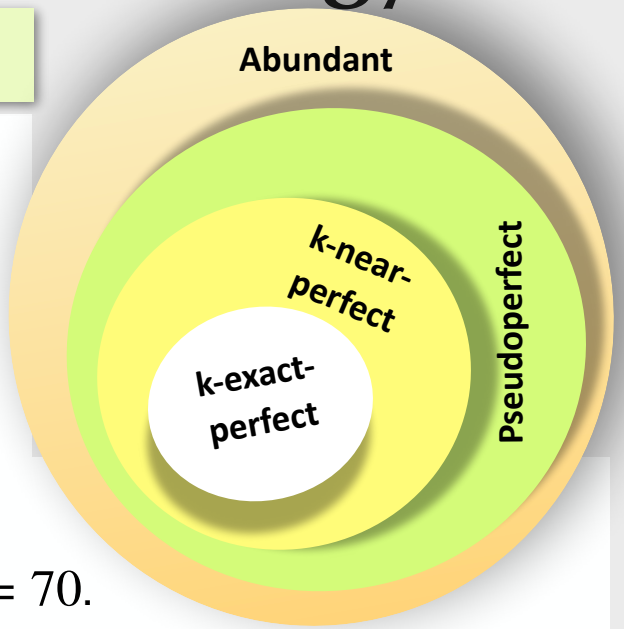
Integer Classification

Let $D_n = \{d \in \mathbb{N} \mid d \mid n\}$ be the set of divisors of integer n. The ***sum-of divisors*** is the sum of all distinct divisors of n, and is given by $\sigma(n) = \sum d$.

Then, ***abundance of n*** is defined as $A(n) = \sigma(n) - 2n$. Known classifications:

- ***Perfect***: $A(n) = 0$,
- ***Deficient***: $A(n) < 0$,
- ***Abundant***: $A(n) > 0$.

- Abundant integers are further classified as:
  - ***Pseudoperfect***: $A(n) = \sum_{g \neq n} g, g \in G \subsetneq D_n$,
  - Otherwise, ***Weird***, eg. $n = 70$.



k-Near-Perfect Number (k-npn) & k-Exact-Perfect Number (k-epn)

- A pseudoperfect number n is a ***k-near-perfect number (k-npn)*** if it has a set $R_{n,r}$ of r distinct proper divisors, called ***redundant divisors,*** that sum to $A(n)$, where $0 < r \leq k$ [2]. eg. $n = 24$ is 4-npn.
  - k-npn: $A(n) = \sum_{g \neq n} g, g \in G \subsetneq D_n, 0 < |G| \leq k$,
  - $R_{n,|G|}$ is the ***redundant set*** of size $|G|$ for n [3].
- A k-npn n is a ***k-exact-perfect number (k-epn)*** if it has a redundant set $R_{n,k}$ of size $\boldsymbol{k}$ [4].

  eg. $n = 24$ is 1-epn, 2-epn, 3-epn and 4-epn.
  - k-epn: $A(n) = \sum_{g \neq n} g, g \in G \subsetneq D_n, |G| = k$.

Framework: Proposed Properties

Redundancy Degree $\rho_k(n)$

Definition:

Let n and k be positive integers. The i^{th} redundant set of n, denoted by $R^i_{n,k}$, has exactly k distinct divisors of n that sum to the abundance $A(n)$. Let,

$$\mathscr{R}_{n,k} = \{R^1_{n,k}, R^2_{n,k}, \ldots, R^m_{n,k}\}$$

denote the complete collection of m distinct size-k redundant sets of n. Then, the ***redundancy degree*** of n with respect to k, denoted as $\rho_k(n)$, is defined as the cardinality of $\mathscr{R}_{n,k}$, that is,

$$\rho_k(n) = |\mathscr{R}_{n,k}| = m$$

In other words,

$$A(n) = \sum_{r \in R^1_{n,k}} r = \sum_{r \in R^2_{n,k}} r = \ldots \sum_{r \in R^m_{n,k}} r.$$

If no redundant set of size k exists, then $\rho_k(n) = 0$.

Proposed Framework

- This novel framework enables us to analyze redundant divisors through new parameters- redundancy degree and reconstructibility.
- Redundancy degree gives insights into combinatorial richness of redundant divisors.

Reconstructibility

Definition:

A k-epn n satisfies ***reconstructibility*** if n can be determined from its size-k redundant sets by a deterministic rule. In this work, we say that n is reconstructible if:

$$\mathrm{lcm}(\cup_{i=1}^{\rho_k(n)} R^i_{n,k}) = n$$

In other words, if n is the least-common-multiple (lcm) of all the divisors of $\mathscr{R}_{n,k}$, then it is ***reconstructible***.

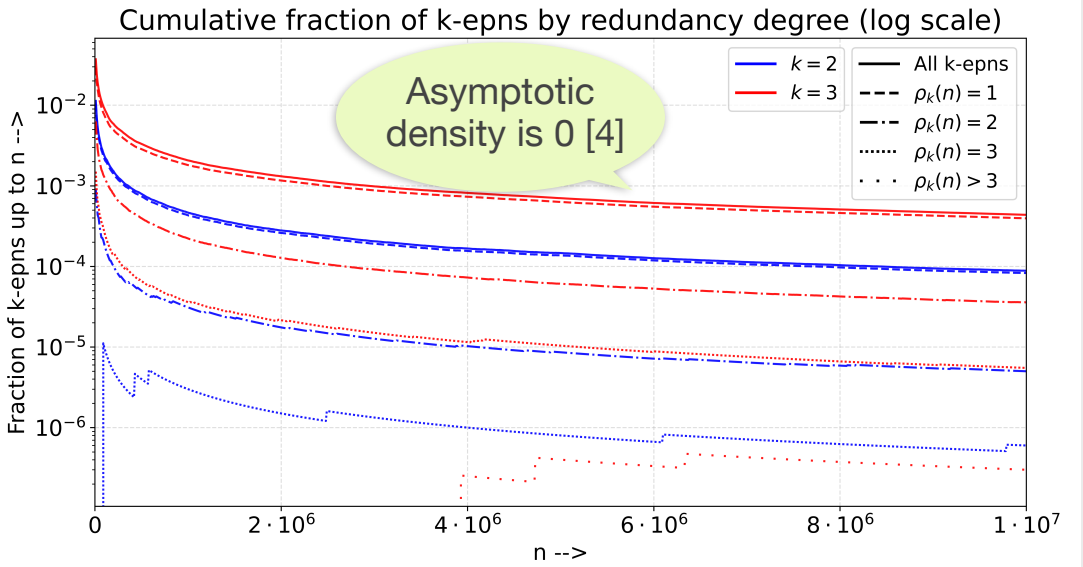
Distinction from Prior Work

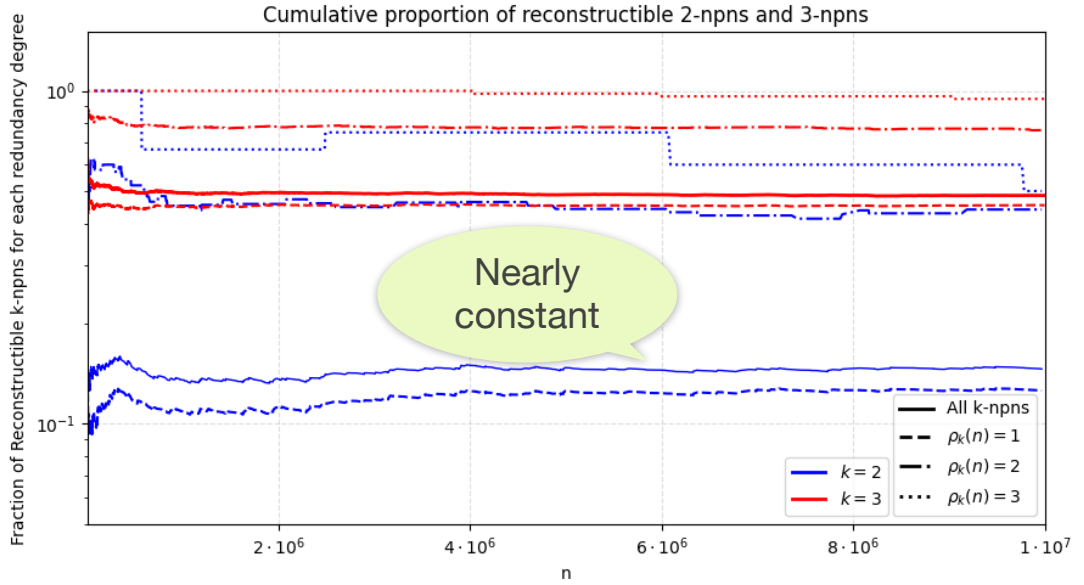
- [2] defined k-npns. [1],[2],[3],[4] classified many families as 1-npns, 2-npns, and k-npns.
- [4] defined k-epns as k-npns with exactly size-k redundant sets.
- [2] and [4] studied the asymptotic behavior of k-npns.
- [2] gave some numbers that are both 1-npn and k-npn.
- This work extends the classification beyond the traditional definitions and looks for patterns within k -epns that show a multiplicity of redundant sets and reconstructibility.

Findings: $\rho_k(n)$, Reconstructibility for $k = 2,3$

Results from Computational Search in Natural Numbers

Brute-force search in python [15] up to $n = 10^7$ to observe cumulative density of 2-epns and 3-epns with and proportion of reconstructible integers for each observed redundancy degree.





- 3-epns have greater density than 2-epns. However, for both 2-epns and 3-epns, the density approaches zero for all redundancy degrees. This is expected as [2] showed the asymptotic density of 0 for small k.
- Found $\rho_2(n) \leq 3, \rho_3(n) \leq 5$ within search range.
- As $\rho_k(n)$ increases, the density of 2-epns and 3-epns decreases. So, numbers with higher multiplicity of redundant sets are rarer. —> novel finding
- As $\rho_k(n)$ increases, the likelihood of reconstructibility increases for 2-epns, 3-epns. The proportion of reconstructible integers to total number of 2-epns or 3-epns is nearly constant. —> novel finding

Discovery of a rich structure:

$n = 2^a \cdot p^2$

$p = 2^a - 1$ (Mersenne prime)

$A(2^a \cdot p^2) = 2^{2a} + 2^a - 1$

$R^1_{n,3}$	1	$2^a p$	$2p$
$R^2_{n,3}$	$2^a p$	p	2^a
$R^3_{n,3}$	$2p$	2^a	p^2

$\boldsymbol{\rho_3(n) = 3}$

Semi-magic Square: rows, columns sum to $A(n)$.

Findings: k-epns in $n = 2^a p$

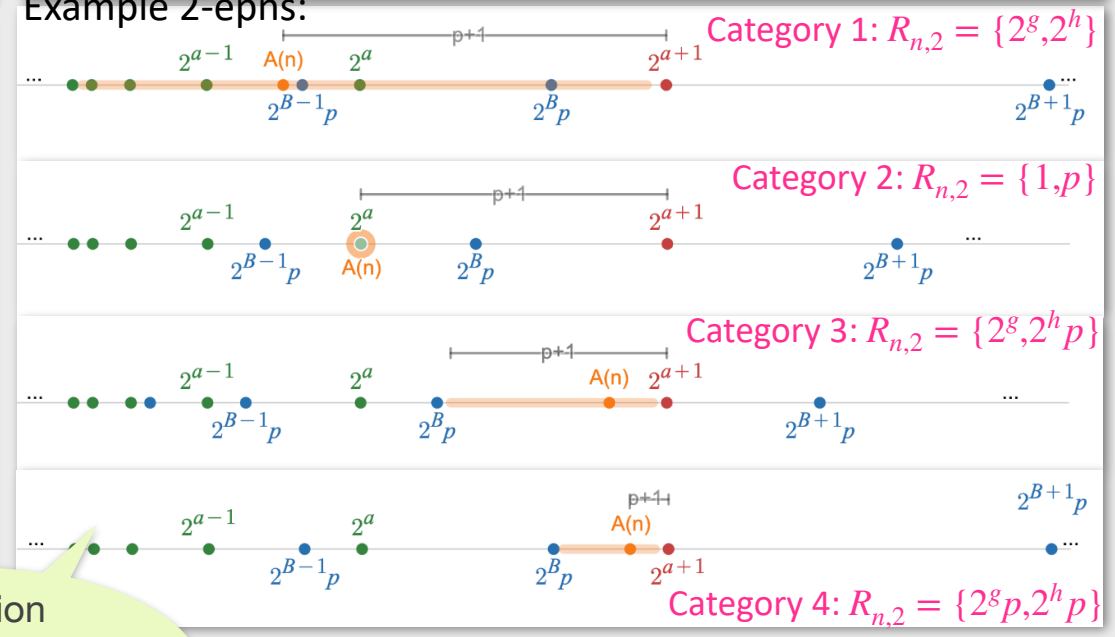
$$A(2^a \cdot p) = \sigma(2^a \cdot p) - 2 \cdot 2^a \cdot p$$
$$= 2^{a+1} - p - 1$$

Analytical Approach: Methodology

- Classify n into categories [1] based on the structure of the redundant divisors.
- Solution is at the intersection of the categories. Solve system of simultaneous Diophantine equations up to 7 unknowns (~31 proofs).
- Use casework. Helper tools reduce casework load:
  - define bounds on $A(n)$,
  - divisor $2^B p$ such that $2^a < 2^B p < 2^{a+1}$,
  - $A'(n)$: $A(n)$ normalized by upper bound $2^{a+1} - 1$.
- Show reconstructibility from LCM of redundant divisors.

Helper Tool#1: Bounds on $A(n)$ for 2-epn

Example 2-epns:



Solution does not exists unless the ranges of $A(n)$ overlap

Computational Search

- Search for redundancy degree, redundant sets and reconstructibility in 2-epns and 3-epns up to $n = 10^{10}$.
- Improved speed by explicitly constructing integers $n = 2^a p$ using Eratosthenes Sieve instead of searching sequentially.
- Reduced search time from 8+ days for sequential search to 10s of seconds with new approach for on Apple M3 Pro MacBook.

Helper Tool #2: Divisor $2^B p$

Table 8: 3-epn: Restriction of h in the redundant set and B

Category #	h	Other Parameters
1	$< a$	
2	1	$g < a$
	2	$g = a$
3	B	$g < a$
	$B-1$	$g = a, f < a-1$
	$B-2$	$g = a, f = a-1$
4	B	–
5	$B, B-1$	$f < a$
	$B-1$	$f = a$
6	B	–

The largest redundant divisor $2^h p$ which is a multiple of p is constrained

In some cases other divisors are also constrained by B

Findings: Sample Proof

Multiple Redundant Sets of Category 3 for 2-epns of the form $n = 2^a p$

Proposition. *2-epns of the form $n = 2^{2b}(2^b - 1)$ have two distinct redundant sets given by $\{2^{2b}, 2^b p\}$ and $\{2^b, 2^{b+1} p\}$. Other 2-epns of in Category 3 have only one redundant set of the form $\{2^g, 2^h p\}$.*

Proof. Assume that n is a 2-epn with two redundant sets $R^1_{n,2} = \{2^g, 2^h p\}$ and $R^2_{n,2} = \{2^u, 2^v p\}$, where $0 < g \leq a, 0 < h < a$ and $0 < u \leq a, 0 < v < a$, $g \neq u$ and $h \neq v$. Since both sets are redundant sets for a 2-epn, they sum to the same abundance. Hence,

$$2^g + 2^h p = 2^u + 2^v p. \tag{*}$$

WLOG, assume that $g > u$. $h < v$ because p is positive. Factoring out powers of 2 from $(*)$ we get,

$$p = \frac{2^u}{2^h} \cdot \frac{2^{g-u} - 1}{2^{v-h} - 1}$$

Since p is odd, the factor $2^u/2^h$ must equal 1, so $u = h$. Substituting for u, we have

$$p = \frac{2^{g-h} - 1}{2^{v-h} - 1} \tag{**}$$

From $(**)$, we infer that $g > v$. Next, we substitute for p of Category 3 into $(**)$ and get

$$\frac{2^{a+1} - 2^g - 1}{2^h + 1} = \frac{2^{g-h} - 1}{2^{v-h} - 1}$$

After clearing denominators and simplifying, we get:

$$2^{a+v-h+1} + 2^h + 2 = 2^{a+1} + 2^{g+v-h} + 2^{g-h} + 2^{v-h}$$

Continued ....

Findings: Sample Proof Continued

On the LHS, the exponents of the three terms are related by $a+v-h+1 > h \geq 1$. On the RHS, the exponents of the four terms are related by $g+v-h > g-h > v-h$ and $a+1 > g-h$. To solve for g, v, h, a, we examine the possible values of h.

- $\boldsymbol{h > 1}$: All the terms on LHS are distinct. Matching the smallest powers of 2 gives $v = h+1$. Then matching the largest powers gives $a = g$. Matching the middle powers yields $g = 2h$.

  Consolidating the results, we have: $g = a, h = \frac{a}{2}, u = \frac{a}{2}, v = \frac{a}{2}+1$, where a has to be even for a solution and $p = 2^{\frac{a}{2}} - 1$. For ease of notation, let $a = 2b$. Then, we have the only solution $g = 2b, h = b, u = b, v = b+1$ and $p = 2^b - 1$. Thus, $n = 2^{2b}(2^b - 1)$ has two distinct redundant sets:

$$R^1_{n,2} = \{2^{2b},\, 2^b p\}, \qquad R^2_{n,2} = \{2^b,\, 2^{b+1} p\}.$$

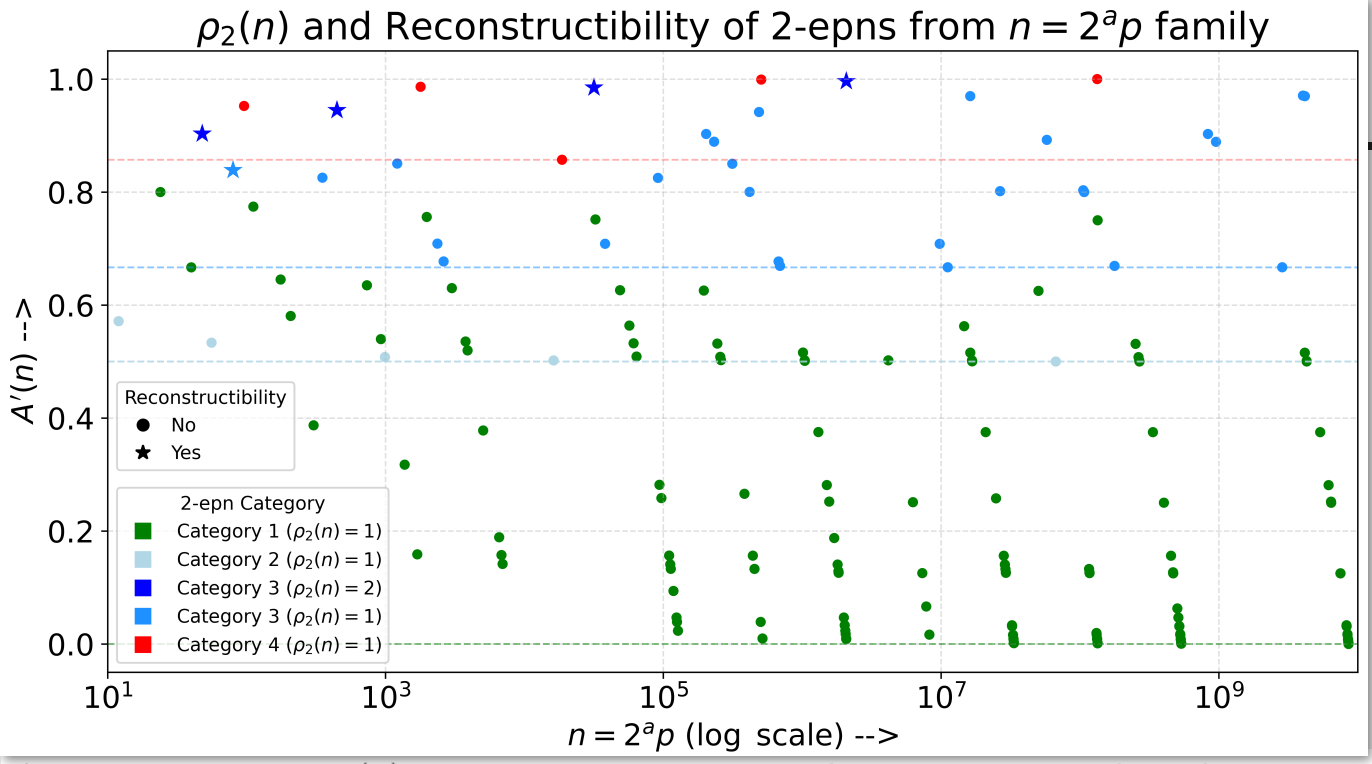
- $\boldsymbol{h = 1}$: We have $2^{a+v} + 2^2 = 2^{a+1} + 2^{g+v-1} + 2^{g-1} + 2^{v-1}$. Now we have a sum of two distinct powers of 2 on the LHS. Since 2^{v-1} is the smallest term on the RHS, we equate it to the smallest term 2^2 on the LHS and get $v - 1 = 2 \implies v = 3$. Of the remaining three terms on the RHS, 2^{g-1} is the smallest and does not combine with either of the other two terms to generate a single power of 2. So, this system does not have a solution when $h = 1$.

For 2-epns of Category 3 which are of the form $n = 2^{2b} \cdot (2^b - 1)$, the redundancy-degree is $\rho_2(n) = 2$ with two distinct redundant sets given by $\{2^{2b}, 2^b p\}$ and $\{2^b, 2^{b+1} p\}$. For other integers of Category 3, the redundancy-degree is $\rho_2(n) = 1$. $\square$

Findings: Results

Analysis, Computational Search for 2-epns ($n = 2^a p$)

- Exactly one subfamily of Mersenne prime based construction has $\rho_k(n) = 2$. It is also reconstructible.
- All others 2-epns have $\rho_k(n) = 1$ and all except $n = 2^4 \cdot 5$ are not reconstructible.
- Category-1 is most frequent due to the large range of $A(n)$.
- On the log-log plot, each waterfall curves corresponds to a specific value of a.
- There is agreement between analytical results and computational search results.



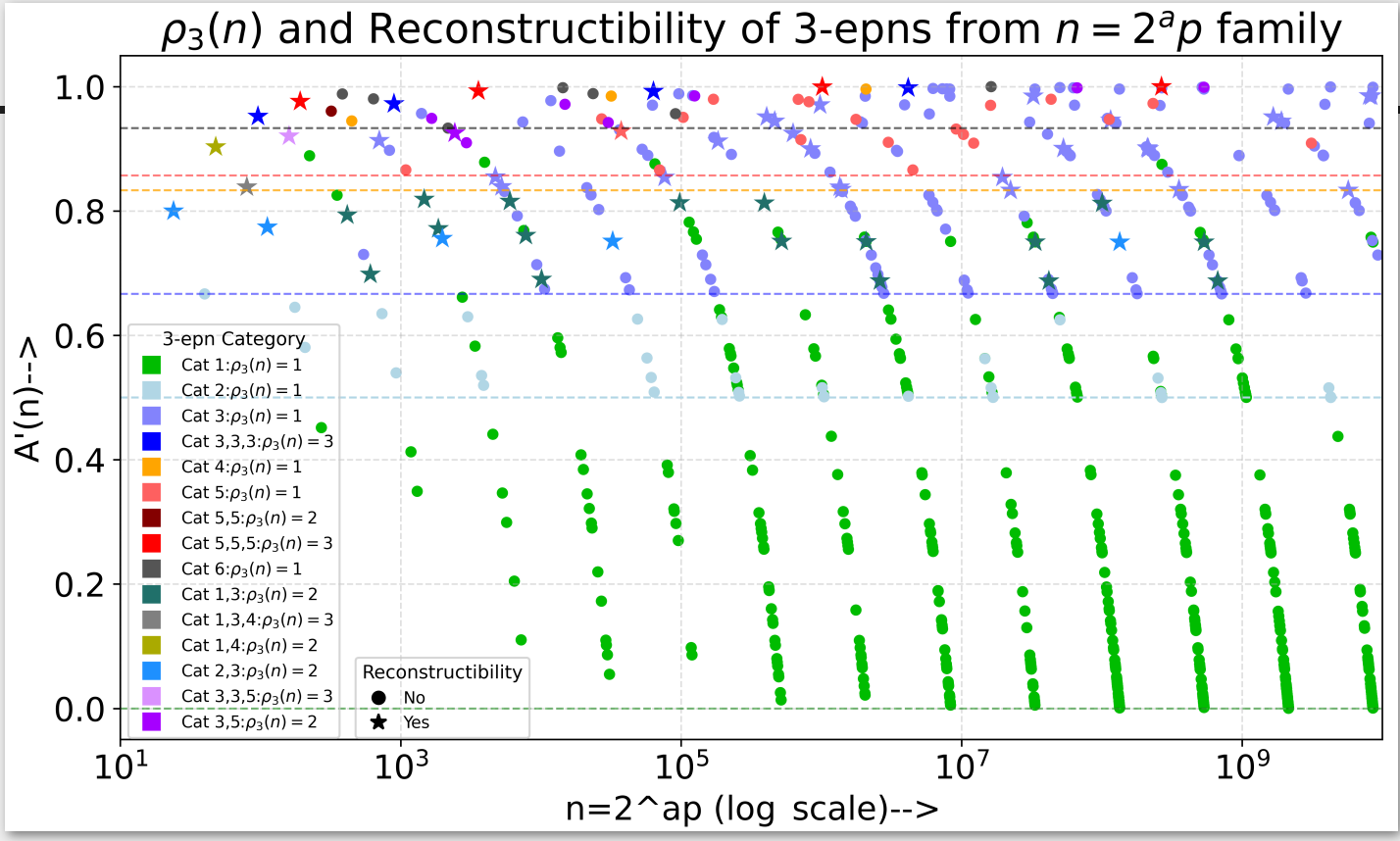
Analytical results: $\rho_2(n)$ and Reconstructilibility of 2-epn numbers of the form $n = 2^a p$.

	Cat.	p		$R^i_{n,2}$	$\rho_2(n)$	Rec.
●	1	$2^{a+1} - 2^g - 2^h - 1$	$0 < g < h \leq a$	$\{2^g, 2^h\}$	1	No
●	2	$2^a - 1$	$p = 2^a - 1$, $a \geq 2$, $g = 0, h = 0$	$\{1, 2^a - 1\}$	1	No
★	3	$\frac{2^{a+1} - 2^g - 1}{1 + 2^h}$	$a = g = 2b, h = b$ $p = 2^b - 1$.	$\{2^{2b}, 2^b p\}$, $\{2^b, 2^{b+1} p\}$	2	Yes
★			$a = 4, g = 4, h = 1$ $p = 5, n = 2^4 \cdot 5$	$\{2^1 \cdot 5, 2^4\}$	1	Yes
●			Others, $0 < g \leq a$, $0 < h < a$	$\{2^g, 2^h p\}$	1	No
●	4	$\frac{2^{a+1} - 1}{1 + 2^g + 2^h}$	$0 < g < h < a$	$\{2^g p, 2^h p\}$	1	No

Findings: Results

Analysis, Computational Search for 3-epns ($n = 2^a p$)

- There are subfamilies using both Mersenne primes and other primes that conjectured to be infinitely many.
- Some subfamilies have multiple distinct redundant sets of the same category.
- Some subfamilies have redundant sets of different categories.
- Some subfamilies satisfy reconstructibility.
- Category-1 is most frequent due to the large range of $A(n)$.
- The category for which all redundant divisors are multiples of p is not reconstructible for any k (Category-6 in case of 3-epn).
- Only snapshot of table of analytical results is shown. See full analytical results in [11].



Selected Examples

	Cat.	Solutions	$R^i_{n,3}$	$\rho_3(n)$	Rec.
★	3	$p = \frac{2^{2b+2} - 2^f - 2^g - 1}{1 + 2^h} = 2^b - 1,\ n = 2^{2b+1} \cdot p$	$\{2^b, 2^{b+1}, 2^{b+2}p\}$, $\{2^b, 2^{2b+1}, 2^{b+1}p\}$, $\{2^{2b}, 2^{2b+1}, 2^b p\}$	3	Yes
★		$p = \frac{2^a - 2^f - 1}{2^h + 1}, 0 < f, h < a.$	$\{2^f, 2^a, 2^h p\}$	1	Yes
●		$p = \frac{2^{a+1} - 2^f - 2^g - 1}{2^h + 1}, 0 < f < g < a, 0 < h < a.$	$\{2^f, 2^g, 2^h p\}$	1	No
★	1,3	$p = 2^{a+1} - 2^f - 2^g - 2^h - 1 = \frac{2^{a+1} - 2^u - 2^v - 1}{1 + 2^w}.$	$\{2, 2^{a-1}, 2^a\}$, $\{2^3, 2^{a-1}, 2p\}$	2	Yes
★		$p = 2^{a+1} - 2^f - 2^g - 2^h - 1 = \frac{2^{a+1} - 2^u - 2^v - 1}{1 + 2^w}.$	$\{2^{a-3}, 2^{a-2}, 2^a\}$, $\{2, 2^{a-3}, 2p\}$	2	Yes
★		$p = 2^{a+1} - 2^f - 2^g - 2^h - 1 = \frac{2^{a+1} - 2^u - 2^v - 1}{1 + 2^w}.$	$\{2^{a-3}, 2^{a-1}, 2^a\}$, $\{2^2, 2^{a-3}, 2^2 p\}$	2	Yes

Conclusions

Assessments

- **Quantifying multiplicity**: Defined *Redundancy Degree*, $\rho_k(n)$, for k-epn n to quantify multiplicity of size-k redundant sets.
- **Information Capture:** Defined *Reconstructibility* to evaluate if the information of n is preserved in its redundant sets.
- **Closed-form Characterization:** Fully enumerated redundancy degree, reconstructibility, structure of redundancy set of all categories of 2-epns and 3-epns in the form $n = 2^a p$.
- **Asymptotic density of 0**: In agreement with [2] and [4].
- Hypotheses formed at the start of research are proven to be correct.

Applications

- Divisor-sum structures are used in number search, constrained partition theory, subset sum problems, etc.
- Cryptography and distributed authentication techniques also rely on number structures for secure decoding.

Limitations

- *Reconstructibility* is through an LCM operation here. However, reconstruction may be possible by other methods like local search or inference from other divisors. These methods can change some of the conclusions here.
- Solving 7-variable Diophantines was hard due to extensive casework. Investigate elegant solutions for complex families.

Advancement Over Prior Work

- **Higher-resolution classification** of abundant numbers through *redundancy degree*.
- New way to **reconstruct integer** form partial information in multiple redundant sets.
  - **New Perspective**: Multiple redundant sets are like distributed fingerprints of a number. Abundance is like a public key, reconstructibility checks distributed private keys.
- Discovery of **new divisor structures** in 2-epns and 3-epns for $n = 2^a p$.

Further Research

- In this research, only size-k sets are considered in reconstruction. Investigate what happens if all sizes can be used? Do most numbers becomes reconstructible?
- Investigate if reconstructibility if unique for some numbers.
- Derive redundancy degrees for other families of numbers, especially for larger k, as we expect to find more complex patterns of divisors sums.
- Find bounds on density for different redundancy degrees.

References/Supplemental Information

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