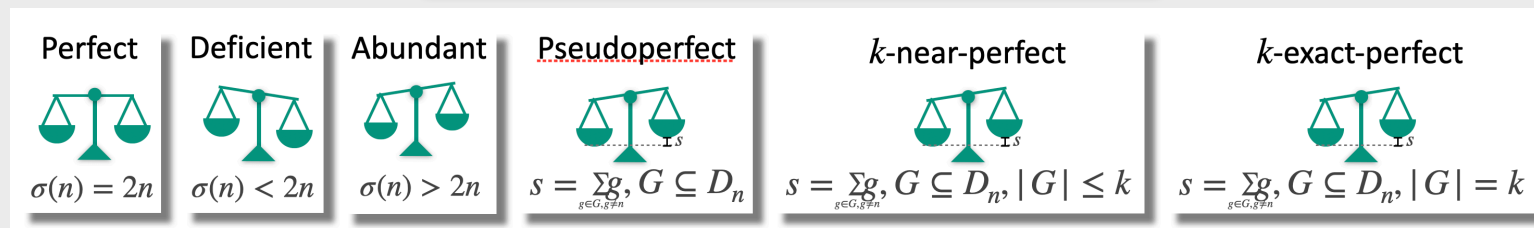


Redundancy Degree and Reconstructibility in k -Exact-Perfect Numbers

- Pranav Krishnapuram



- ◆ This number-theoretic research introduces a novel framework to classify a k -exact-perfect number (k -epn) by the structure of its *redundant divisors*. We define two properties to classify k -epns:
 - *Redundancy Degree*: counts the number of redundant sets of k redundant divisors.
 - *Reconstructibility*: determines whether the least-common-multiple of divisors of all size- k redundant sets equals n .
- ◆ We study these properties for 2-epns and 3-epns and show through computational search that, as redundancy degree increases,
 - the density of k -epns decreases, and
 - the likelihood of reconstructibility increases.
- ◆ Through both analysis and computational search in $n = 2^a p$ for odd prime $p < 2^{a+1}$, we show that,
 - Among 2-epns, exactly one subfamily with Mersenne primes has redundancy degree greater than 1 and satisfies reconstructibility.
 - Among 3-epns, multiple subfamilies have redundancy degrees up to 3 and reconstructibility.
 - All sub-families can be fully expressed through formulas for their redundant divisors.

Introduction

Continuation

None

Work by Others (Literature Survey)

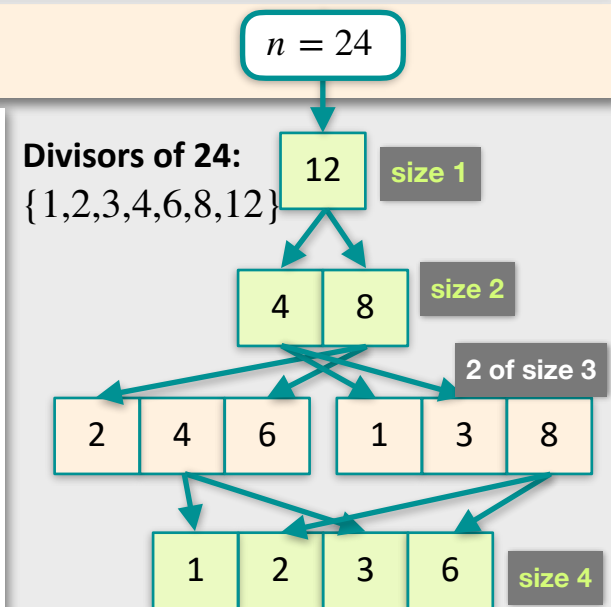
- ◆ [2],[3] defined a *k*-near-perfect-number (*k*-n_{pn}) as a number with at least *k* divisors that sum to the abundance of the number.
 - The *k* divisors are called “redundant divisors” and the set of the *k* redundant divisors are called “redundant set”.
- ◆ [1],[2],[3] studied $n = 2^a p$, $n = 2^a pq$, $n = 2^a p^i$. [2],[3] derived solutions for asymptotic density of *k*-npns.
- ◆ [4] defined *k*-exact-perfect-numbers (*k*-epns) as integers with exactly *k* redundant divisors.

Project Origin

- ◆ I got inspiration for this project from [1], a publication by high schoolers on 2-near-perfect-numbers.
- ◆ Preliminary computational search revealed rich patterns beyond traditional classifications. I found multiple redundant sets of size 2, 3. So, I defined “redundancy degree” to denote the multiplicity of redundant sets.
- ◆ Multiple redundant sets expose more divisors. I wondered how information about the number is embedded in the divisors and if multiplicity affects how the divisors can reconstruct the number. So, I defined “reconstructibility”.

Research Questions

- ◆ **Key questions** based on thought-provoking patterns in redundant sets:
 - Can *n* have multiple redundant sets of size *k*? For which *n* does multiplicity exist? How large is the multiplicity? How does the density of *k*-epns vary with multiplicity.
 - As the multiplicity increases, more divisors of *n* are revealed through the redundant sets. Is there enough information in the redundant sets to reconstruct *n*? Which *n* are reconstructible? How to reconstruct *n*?
- ◆ **Research Focus:** *k* = 2,3 and simple-to-analyze numbers of the form $n = 2^a p$
- ◆ **Hypotheses:**
 - Asymptotic density of numbers with a multiplicity of redundant sets is 0.
 - Numbers with higher multiplicity will be rarer than those with lower multiplicity.
 - Numbers with higher multiplicity will have denser reconstructibility.



Redundant Sets for $n = 24$: One each of sizes 1, 2, 4, two of size 3. Some sets jointly reveal *n*.

Framework: Background and Terminology

Integer Classification

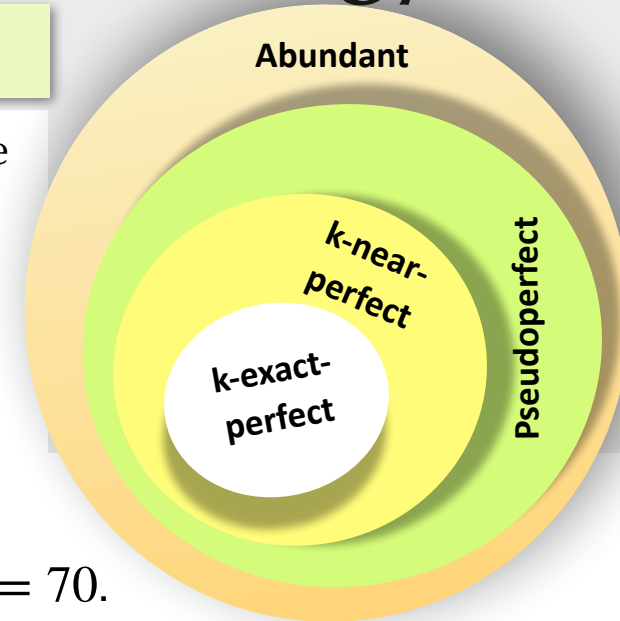
Let $D_n = \{d \in \mathbb{N} \mid d \mid n\}$ be the set of divisors of integer n . The **sum-of divisors** is the sum of all distinct divisors of n , and is given by $\sigma(n) = \sum d$.

Then, **abundance of n** is defined as $A(n) = \sigma(n) - 2n$. Known classifications:

◆ **Perfect**: $A(n) = 0$, ◆ **Deficient**: $A(n) < 0$, ◆ **Abundant**: $A(n) > 0$.

◆ Abundant integers are further classified as:

⊙ **Pseudoperfect**: $A(n) = \sum_{g \neq n} g, g \in G \subsetneq D_n$ ⊙ Otherwise, **Weird**, eg. $n = 70$.



k -Near-Perfect Number (k -npn) & k -Exact-Perfect Number (k -epn)

◆ A pseudoperfect number n is a **k -near-perfect number (k -npn)** if it has a set $R_{n,r}$ of r distinct proper divisors, called **redundant divisors**, that sum to $A(n)$, where $0 < r \leq k$ [2]. eg. $n = 24$ is 4-npn.

⊙ k -npn: $A(n) = \sum_{g \neq n} g, g \in G \subsetneq D_n, 0 < |G| \leq k$,

⊙ $R_{n,|G|}$ is the **redundant set** of size $|G|$ for n [3].

◆ A k -npn n is a **k -exact-perfect number (k -epn)** if it has a redundant set $R_{n,k}$ of size k [4].

eg. $n = 24$ is 1-epn, 2-epn, 3-epn and 4-epn.

⊙ k -epn: $A(n) = \sum_{g \neq n} g, g \in G \subsetneq D_n, |G| = k$.

Framework: Proposed Properties

Redundancy Degree $\rho_k(n)$

Definition:

Let n and k be positive integers. The i^{th} redundant set of n , denoted by $R_{n,k}^i$, has exactly k distinct divisors of n that sum to the abundance $A(n)$. Let,

$$\mathcal{R}_{n,k} = \{R_{n,k}^1, R_{n,k}^2, \dots, R_{n,k}^m\}$$

denote the complete collection of m distinct size- k redundant sets of n . Then, the **redundancy degree** of n with respect to k , denoted as $\rho_k(n)$, is defined as the cardinality of $\mathcal{R}_{n,k}$ that is,

$$\rho_k(n) = |\mathcal{R}_{n,k}| = m$$

In other words,

$$A(n) = \sum_{r \in R_{n,k}^1} r = \sum_{r \in R_{n,k}^2} r = \dots = \sum_{r \in R_{n,k}^m} r.$$

If no redundant set of size k exists, then $\rho_k(n) = 0$.

Proposed Framework

- ◆ This novel framework enables us to analyze redundant divisors through new parameters- redundancy degree and reconstructibility.
- ◆ Redundancy degree gives insights into combinatorial richness of redundant divisors.

Reconstructibility

Definition:

A k -epn n satisfies **reconstructibility** if n can be determined from its size- k redundant sets by a deterministic rule. In this work, we say that n is reconstructible if:

$$\text{lcm}(\cup_{i=1}^{\rho_k(n)} R_{n,k}^i) = n$$

In other words, if n is the least-common-multiple (lcm) of all the divisors of $\mathcal{R}_{n,k}$, then it is **reconstructible**.

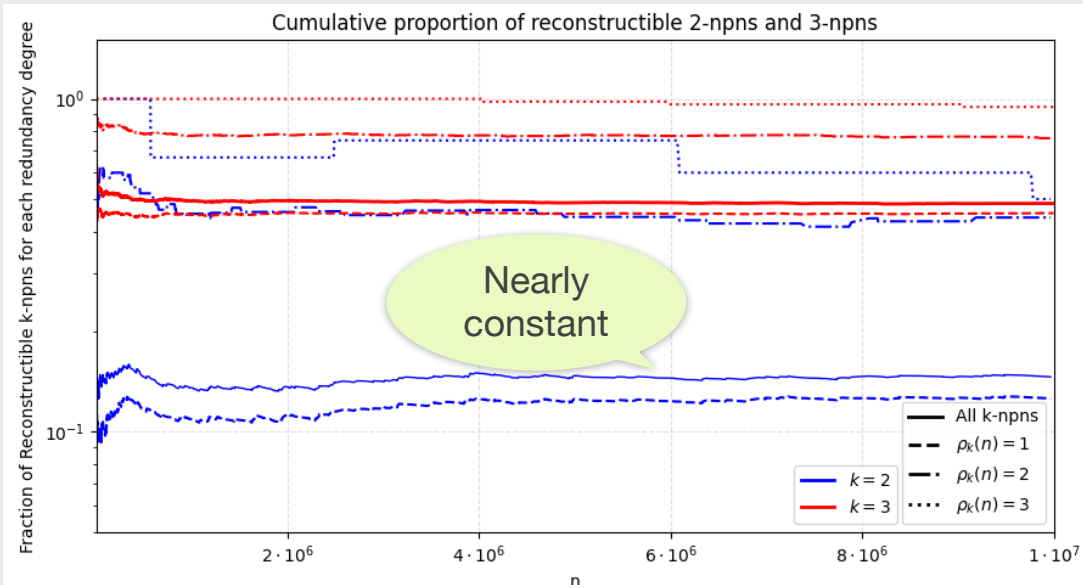
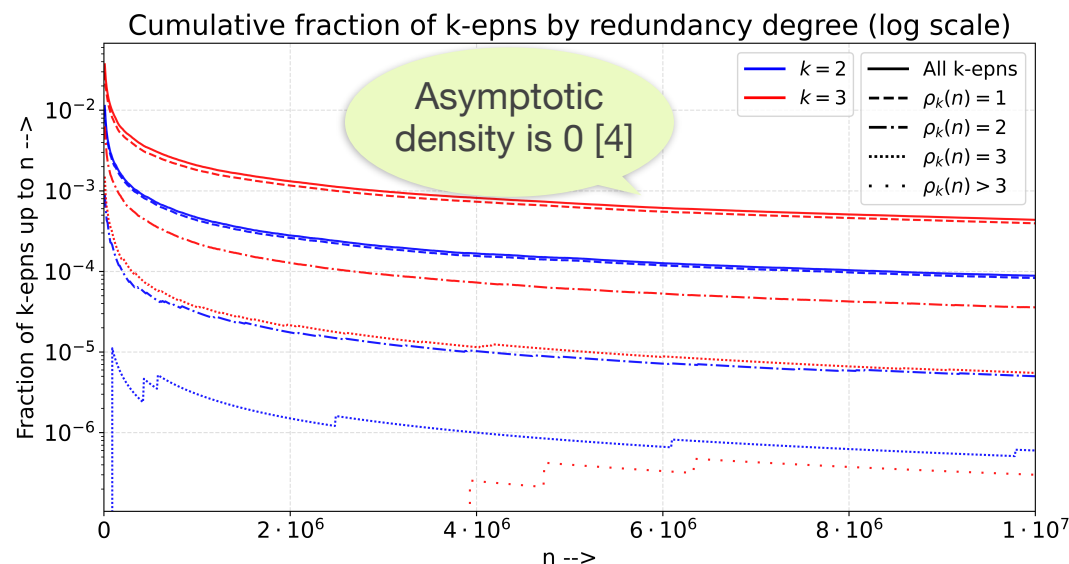
Distinction from Prior Work

- ◆ [2] defined k -npns. [1],[2],[3],[4] classified many families as 1-npns, 2-npns, and k -npns.
- ◆ [4] defined k -epns as k -npns with exactly size- k redundant sets.
- ◆ [2] and [4] studied the asymptotic behavior of k -npns.
- ◆ [2] gave some numbers that are both 1-npn and k -npn.
- ◆ This work extends the classification beyond the traditional definitions and looks for patterns within k -epns that show a multiplicity of redundant sets and reconstructibility.

Findings: $\rho_k(n)$, Reconstructibility for $k = 2, 3$

Results from Computational Search in Natural Numbers

Brute-force search in python [15] up to $n = 10^7$ to observe cumulative density of 2-epns and 3-epns with and proportion of reconstructible integers for each observed redundancy degree.



- 3-epns have greater density than 2-epns. However, for both 2-epns and 3-epns, the density approaches zero for all redundancy degrees. This is expected as [2] showed the asymptotic density of 0 for small k .
- Found $\rho_2(n) \leq 3, \rho_3(n) \leq 5$ within search range.
- As $\rho_k(n)$ increases, the density of 2-epns and 3-epns decreases. So, numbers with higher multiplicity of redundant sets are rarer. —> novel finding
- As $\rho_k(n)$ increases, the likelihood of reconstructibility increases for 2-epns, 3-epns. The proportion of reconstructible integers to total number of 2-epns or 3-epns is nearly constant. —> novel finding

Discovery of a rich structure:

$$n = 2^a \cdot p^2$$

$$p = 2^a - 1 \text{ (Mersenne prime)}$$

$$A(2^a \cdot p^2) = 2^{2a} + 2^a - 1$$

$R_{n,3}^1$	1	$2^a p$	$2p$
$R_{n,3}^2$	$2^a p$	p	2^a
$R_{n,3}^3$	$2p$	2^a	p^2

$$\rho_3(n) = 3$$

Semi-magic Square: rows, columns sum to $A(n)$.

Findings: k -epns in $n = 2^a p$

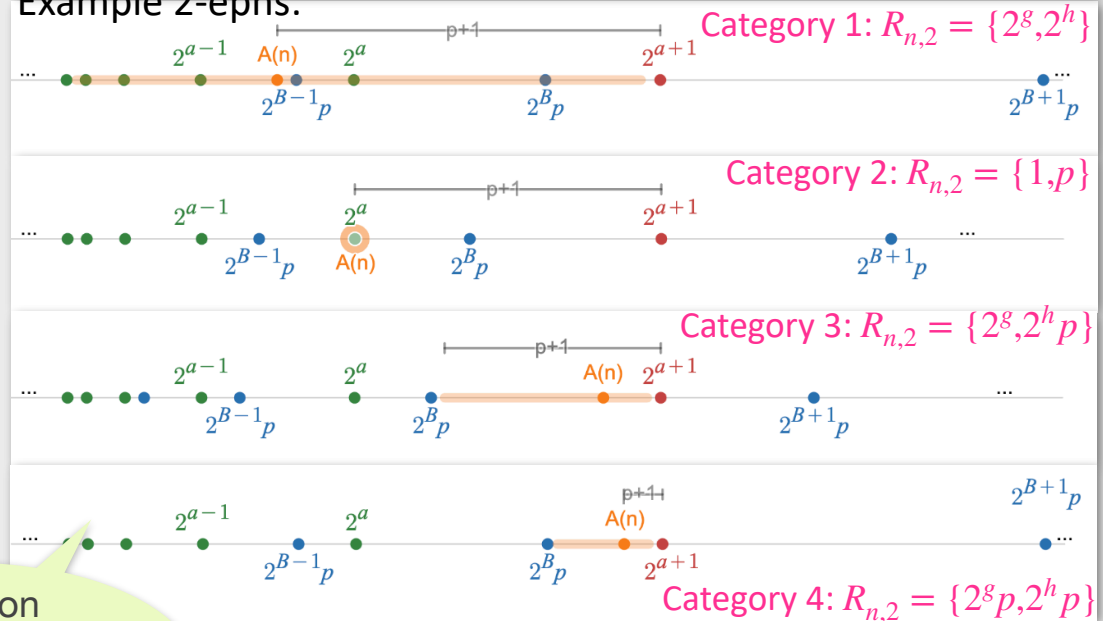
$$\begin{aligned} A(2^a \cdot p) &= \sigma(2^a \cdot p) - 2 \cdot 2^a \cdot p \\ &= 2^{a+1} - p - 1 \end{aligned}$$

Analytical Approach: Methodology

- ◆ Classify n into categories [1] based on the structure of the redundant divisors.
- ◆ Solution is at the intersection of the categories. Solve system of simultaneous Diophantine equations up to 7 unknowns (~31 proofs).
- ◆ Use casework. Helper tools reduce casework load:
 - define bounds on $A(n)$,
 - divisor $2^B p$ such that $2^a < 2^B p < 2^{a+1}$,
 - $A'(n)$: $A(n)$ normalized by upper bound $2^{a+1} - 1$.
- ◆ Show reconstructibility from LCM of redundant divisors.

Helper Tool#1: Bounds on $A(n)$ for 2-epn

Example 2-epns:



Solution does not exist unless the ranges of $A(n)$ overlap

Computational Search

- ◆ Search for redundancy degree, redundant sets and reconstructibility in 2-epns and 3-epns up to $n = 10^{10}$.
- ◆ Improved speed by explicitly constructing integers $n = 2^a p$ using Eratosthenes Sieve instead of searching sequentially.
- ◆ Reduced search time from 8+ days for sequential search to 10s of seconds with new approach for on Apple M3 Pro MacBook.

Helper Tool #2: Divisor $2^B p$

Table 8: 3-epn: Restriction of h in the redundant set and B

Category #	h	Other Parameters
1	$< a$	
2	1	$g < a$
	2	$g = a$
3	B	$g < a$
	$B - 1$	$g = a, f < a - 1$
	$B - 2$	$g = a, f = a - 1$
4	B	–
5	$B, B - 1$	$f < a$
	$B - 1$	$f = a$
6	B	–

The largest redundant divisor $2^h p$ which is a multiple of p is constrained

In some cases other divisors are also constrained by B

Findings: Sample Proof

Multiple Redundant Sets of Category 3 for 2-epns of the form $n = 2^a p$

Proposition. 2-epns of the form $n = 2^{2b}(2^b - 1)$ have two distinct redundant sets given by $\{2^{2b}, 2^b p\}$ and $\{2^b, 2^{b+1} p\}$. Other 2-epns of in Category 3 have only one redundant set of the form $\{2^g, 2^h p\}$.

Proof. Assume that n is a 2-epn with two redundant sets $R_{n,2}^1 = \{2^g, 2^h p\}$ and $R_{n,2}^2 = \{2^u, 2^v p\}$, where $0 < g \leq a, 0 < h < a$ and $0 < u \leq a, 0 < v < a, g \neq u$ and $h \neq v$. Since both sets are redundant sets for a 2-epn, they sum to the same abundance. Hence,

$$2^g + 2^h p = 2^u + 2^v p. \quad (*)$$

WLOG, assume that $g > u$. $h < v$ because p is positive. Factoring out powers of 2 from $(*)$ we get,

$$p = \frac{2^u}{2^h} \cdot \frac{2^{g-u} - 1}{2^{v-h} - 1}$$

Since p is odd, the factor $2^u/2^h$ must equal 1, so $u = h$. Substituting for u , we have

$$p = \frac{2^{g-h} - 1}{2^{v-h} - 1} \quad (**)$$

From $(**)$, we infer that $g > v$. Next, we substitute for p of Category 3 into $(**)$ and get

$$\frac{2^{a+1} - 2^g - 1}{2^h + 1} = \frac{2^{g-h} - 1}{2^{v-h} - 1}$$

After clearing denominators and simplifying, we get:

$$2^{a+v-h+1} + 2^h + 2 = 2^{a+1} + 2^{g+v-h} + 2^{g-h} + 2^{v-h}$$

Continued

Findings: Sample Proof Continued

On the LHS, the exponents of the three terms are related by $a + v - h + 1 > h \geq 1$. On the RHS, the exponents of the four terms are related by $g + v - h > g - h > v - h$ and $a + 1 > g - h$. To solve for g, v, h, a , we examine the possible values of h .

- **$h > 1$:** All the terms on LHS are distinct. Matching the smallest powers of 2 gives $v = h + 1$. Then matching the largest powers gives $a = g$. Matching the middle powers yields $g = 2h$.

Consolidating the results, we have: $g = a, h = \frac{a}{2}, u = \frac{a}{2}, v = \frac{a}{2} + 1$, where a has to be even for a solution and $p = 2^{\frac{a}{2}} - 1$. For ease of notation, let $a = 2b$. Then, we have the only solution $g = 2b, h = b, u = b, v = b + 1$ and $p = 2^b - 1$. Thus, $n = 2^{2b}(2^b - 1)$ has two distinct redundant sets:

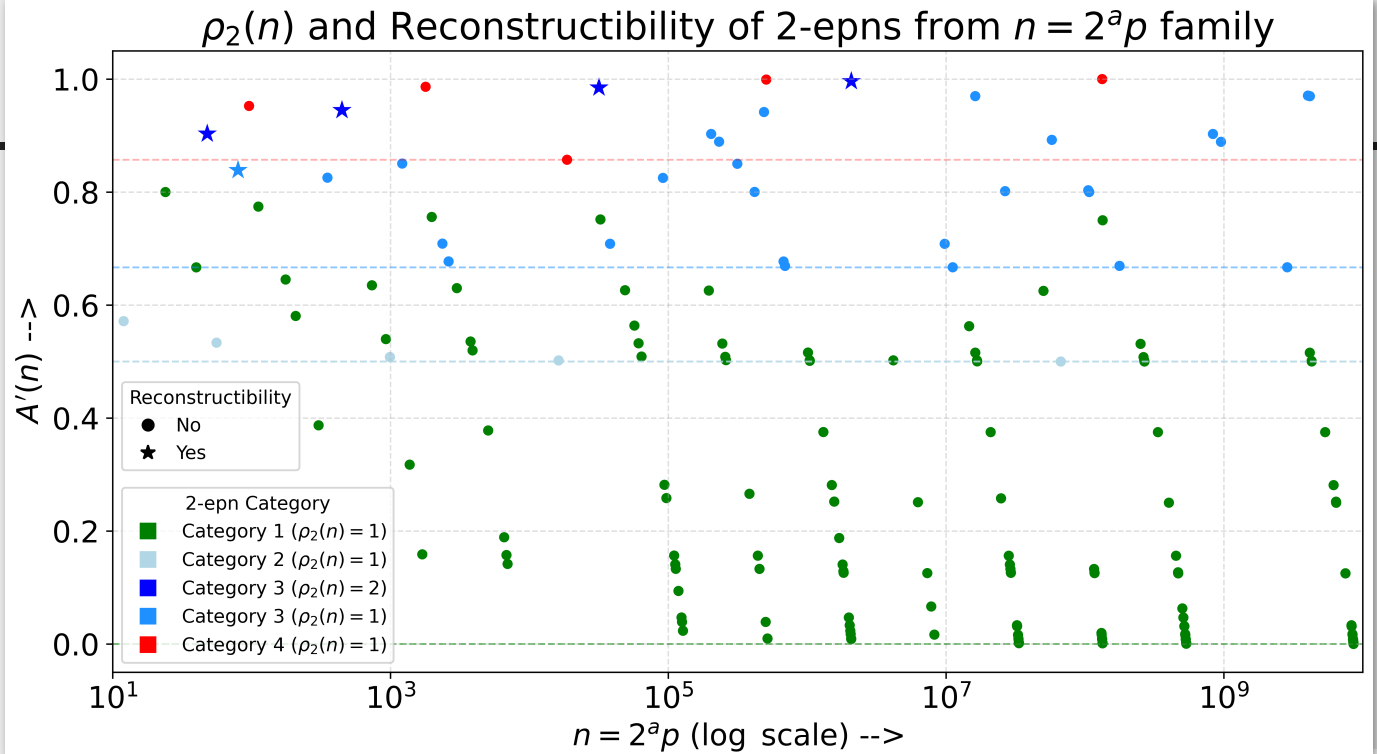
$$R_{n,2}^1 = \{2^{2b}, 2^b p\}, \quad R_{n,2}^2 = \{2^b, 2^{b+1} p\}.$$

- **$h = 1$:** We have $2^{a+v} + 2^2 = 2^{a+1} + 2^{g+v-1} + 2^{g-1} + 2^{v-1}$. Now we have a sum of two distinct powers of 2 on the LHS. Since 2^{v-1} is the smallest term on the RHS, we equate it to the smallest term 2^2 on the LHS and get $v - 1 = 2 \implies v = 3$. Of the remaining three terms on the RHS, 2^{g-1} is the smallest and does not combine with either of the other two terms to generate a single power of 2. So, this system does not have a solution when $h = 1$.

For 2-epns of Category 3 which are of the form $n = 2^{2b} \cdot (2^b - 1)$, the redundancy-degree is $\rho_2(n) = 2$ with two distinct redundant sets given by $\{2^{2b}, 2^b p\}$ and $\{2^b, 2^{b+1} p\}$. For other integers of Category 3, the redundancy-degree is $\rho_2(n) = 1$. □

Findings: Results

- Exactly one subfamily of Mersenne prime based construction has $\rho_k(n) = 2$. It is also reconstructible.
- All others 2-epns have $\rho_k(n) = 1$ and all except $n = 2^4 \cdot 5$ are not reconstructible.
- Category-1 is most frequent due to the large range of $A(n)$.
- On the log-log plot, each waterfall curves corresponds to a specific value of a .
- There is agreement between analytical results and computational search results.



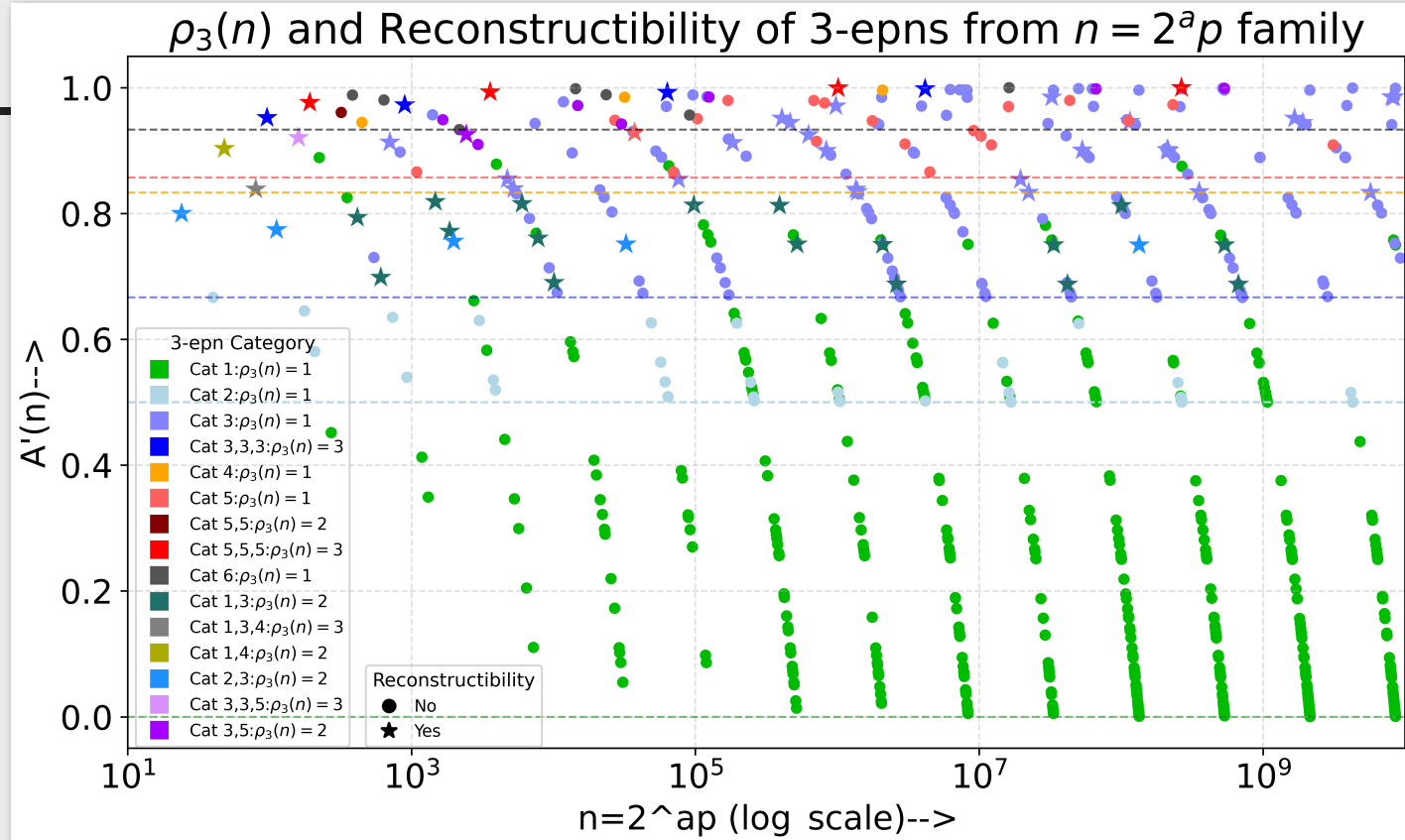
Analytical results: $\rho_2(n)$ and Reconstructibility of 2-epn numbers of the form $n = 2^a p$.

	Cat.	p		$R_{n,2}^i$	$\rho_2(n)$	Rec.
●	1	$2^{a+1} - 2^g - 2^h - 1$	$0 < g < h \leq a$	$\{2^g, 2^h\}$	1	No
●	2	$2^a - 1$	$p = 2^a - 1,$ $a \geq 2, g = 0, h = 0$	$\{1, 2^a - 1\}$	1	No
★	3	$\frac{2^{a+1} - 2^g - 1}{1 + 2^h}$	$a = g = 2b, h = b$ $p = 2^b - 1.$	$\{2^{2b}, 2^b p\},$ $\{2^b, 2^{b+1} p\}$	2	Yes
★			$a = 4, g = 4, h = 1$ $p = 5, n = 2^4 \cdot 5$	$\{2^1 \cdot 5, 2^4\}$	1	Yes
●			Others, $0 < g \leq a,$ $0 < h < a$	$\{2^g, 2^h p\}$	1	No
●	4	$\frac{2^{a+1} - 1}{1 + 2^g + 2^h}$	$0 < g < h < a$	$\{2^g p, 2^h p\}$	1	No

Findings: Results

Analysis, Computational Search for 3-epns ($n = 2^a p$)

- There are subfamilies using both Mersenne primes and other primes that conjectured to be infinitely many.
- Some subfamilies have **multiple distinct redundant sets of the same category**.
- Some subfamilies have **redundant sets of different categories**.
- Some subfamilies satisfy reconstructibility.
- Category-1 is most frequent due to the large range of $A(n)$.
- The category for which all redundant divisors are multiples of p is not reconstructible** for any k (Category-6 in case of 3-epn).
- Only snapshot of table of analytical results is shown. See full analytical results in [11].



	Cat.	Solutions	Selected Examples	$R_{n,3}^i$	$\rho_3(n)$	Rec.
★	3	$p = \frac{2^{2b+2} - 2^f - 2^g - 1}{1 + 2^h} = 2^b - 1, n = 2^{2b+1} \cdot p$	$\{2^b, 2^{b+1}, 2^{b+2}p\},$ $\{2^b, 2^{2b+1}, 2^{b+1}p\},$ $\{2^{2b}, 2^{2b+1}, 2^b p\}$	3	Yes	
★		$p = \frac{2^a - 2^f - 1}{2^h + 1}, 0 < f, h < a.$	$\{2^f, 2^a, 2^h p\}$	1	Yes	
●		$p = \frac{2^{a+1} - 2^f - 2^g - 1}{2^h + 1}, 0 < f < g < a, 0 < h < a.$	$\{2^f, 2^g, 2^h p\}$	1	No	
★	1,3	$p = 2^{a+1} - 2^f - 2^g - 2^h - 1 = \frac{2^{a+1} - 2^u - 2^v - 1}{1 + 2^w}.$	$\{2, 2^{a-1}, 2^a\},$ $\{2^3, 2^{a-1}, 2p\}$	2	Yes	
		$p = 2^{a+1} - 2^f - 2^g - 2^h - 1 = \frac{2^{a+1} - 2^u - 2^v - 1}{1 + 2^w}.$	$\{2^{a-3}, 2^{a-2}, 2^a\},$ $\{2, 2^{a-3}, 2p\}$	2	Yes	
		$p = 2^{a+1} - 2^f - 2^g - 2^h - 1 = \frac{2^{a+1} - 2^u - 2^v - 1}{1 + 2^w}.$	$\{2^{a-3}, 2^{a-1}, 2^a\},$ $\{2^2, 2^{a-3}, 2^2 p\}$	2	Yes	

Conclusions

Assessments

- ◆ **Quantifying multiplicity:** Defined *Redundancy Degree*, $\rho_k(n)$, for k -epn n to quantify multiplicity of size- k redundant sets.
- ◆ **Information Capture:** Defined *Reconstructibility* to evaluate if the information of n is preserved in its redundant sets.
- ◆ **Closed-form Characterization:** Fully enumerated *redundancy degree, reconstructibility, structure of redundancy* set of all categories of 2-epns and 3-epns in the form $n = 2^a p$.
- ◆ **Asymptotic density of 0:** In agreement with [2] and [4].
- ◆ Hypotheses formed at the start of research are proven to be correct.

Applications

- ◆ Divisor-sum structures are used in number search, constrained partition theory, subset sum problems, etc.
- ◆ Cryptography and distributed authentication techniques also rely on number structures for secure decoding.

Limitations

- ◆ *Reconstructibility* is through an LCM operation here. However, reconstruction may be possible by other methods like local search or inference from other divisors. These methods can change some of the conclusions here.
- ◆ Solving 7-variable Diophantines was hard due to extensive casework. Investigate elegant solutions for complex families.

Advancement Over Prior Work

- ◆ **Higher-resolution classification** of abundant numbers through *redundancy degree*.
- ◆ New way to **reconstruct integer** from partial information in multiple redundant sets.
 - **New Perspective:** Multiple redundant sets are like distributed fingerprints of a number. Abundance is like a public key, reconstructibility checks distributed private keys.
- ◆ **Discovery of new divisor structures** in 2-epns and 3-epns for $n = 2^a p$.

Further Research

- ◆ In this research, only size- k sets are considered in reconstruction. Investigate what happens if all sizes can be used? Do most numbers become reconstructible?
- ◆ Investigate if reconstructibility is unique for some numbers.
- ◆ Derive redundancy degrees for other families of numbers, especially for larger k , as we expect to find more complex patterns of divisor sums.
- ◆ Find bounds on density for different redundancy degrees.

Scope of Work

New Work by the Author

- ◆ Defined **Redundancy Degree** and **Reconstructibility** for k -epns [11].
- ◆ Wrote pseudocode for **brute-force** search of all 2-epns and 3-epns up $n = 10^7$ [11].
- ◆ Wrote pseudocode to search all 2-epns and 3-epns of the form $n = 2^a p$ up to $n = 10^{10}$ by explicitly constructing primes through **Eratosthenes sieve** for high speed execution [11].
- ◆ Theoretically analyzed categories of 2-epns and 3-epns of the form $n = 2^a p$ and **solved simultaneous Diophantine** equations for 31 cases (10 for 2-epns and 21 for 3-epns) [11],[12],[13].
 - ⦿ Introduced bounds for $A(n)$, defined special divisor $2^B p$ in the dyadic interval $(2^a, 2^{a+1})$ to bound the highest multiple of p in the redundant set [11], [12], [13].
- ◆ Inferred the **inverse relation** between redundancy degree and density, and the **direct relation** between redundancy degree and reconstructibility [11].
- ◆ **Discovered** previously unknown subfamilies in 2-epns and 3-epns of the form $n = 2^a p$ [11], [14].

References/Supplemental Information

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Journal, Proofs, Logs, Python Code, LLM Transcripts, Correspondence

11. Digital Journal
12. Handwritten Proofs
13. Manuscript Draft
14. Data Logs with remarks
15. Python Code
16. LLM Transcripts
17. Correspondence with Dr. Zelinsky