

Problem: What is the formula of the volume of an  $n$ -d regular tetrahedron with side length  $x$ , with proof?

Note: when a tetrahedron is mentioned, assume it is regular.

Let  $f_n(x)$  represent the function of  $x$  that gives a formula for the  $n$ -d volume of a regular  $n$ -d tetrahedron (an  $(n+1)$  simplex) with side length  $x$ . Let  $x$  be a dummy variable to represent the side of the tetrahedron. Let  $h_n$  be the ratio of the height to  $x$  of an  $n$ -d tetrahedron. Let  $r_n$  be the ratio of the distance from a vertex to the center to the height of an  $n$ -d tetrahedron.

First, understand that, by definition, a tetrahedron with side length 0, no matter its dimension, has  $n$ -d volume 0.

We note that  $f_{n+1}(x) = \int_0^{xh_{n+1}} f_n(\frac{y}{h_{n+1}}) dy$  where  $y$  is a dummy variable and represents the distance from a particular vertex along the corresponding height. Now, we can let  $z = \frac{y}{h_{n+1}}$ , and  $dy = h_{n+1} dz$ . Our previous integral can now be equated to  $h_{n+1} \int_0^x f_n(z) dz$ . But we can again equate  $f_n(z)$ , as  $h_n \int_0^x f_{n-1}(z) dz$ .

If we solve this for lower dimensions, we get that  $f_1(x) = x$ , so  $f_2(x) = h_2 \frac{x^2}{2}$ , and so by induction, with the initial case already shown, we try to get  $f_n(x) = \frac{x^n}{n!} \prod_{i=1}^n h_i$ . We have  $f_{n+1}(x) = h_{n+1} \int_0^x f_n(z) dz$ , and  $f_n(z) = \frac{z^n}{n!} \prod_{i=1}^n h_i$  by induction. The product term is a constant with respect to  $z$ , so we will take it outside the integral. We are left with

$$h_{n+1} \prod_{i=1}^n h_i \int_0^x \frac{z^n}{n!} dz = \frac{x^{n+1}}{(n+1)!} \prod_{i=1}^{n+1} h_i$$

by bringing the  $h_{n+1}$  inside the product and by power rule. With this, induction is finished, so  $f_n(x) = \frac{x^n}{n!} \prod_{i=1}^n h_i$  has been proved. This is great, but leaves us with a different problem. We now have a new term we don't know, the product of the heights. We will now try to find a closed form formula for  $h_n$ .

By the Pythagorean theorem, if we take the right triangle formed by the center of a  $(n-1)$ -d tetrahedron which is the base of an  $n$ -d tetrahedron, one vertex that lays on that base, and the vertex not on the base, we can show that  $x^2 = (xh_n)^2 + (xh_{n-1}r_{n-1})^2$ , so  $1 = h_n^2 + h_{n-1}^2 r_{n-1}^2$ , and therefore

$\sqrt{1 - h_{n-1}^2 r_{n-1}^2} = h_n$ . Again, by the Pythagorean theorem, with points the center of the  $n$ -d tetrahedron, the center of one of its bases, and a vertex on that base, we get  $(xh_n r_n)^2 = (xh_n(1 - r_n))^2 + (xh_{n-1}r_{n-1})^2$  and thus  $h_n^2 r_n^2 = h_n^2 - 2h_n^2 r_n + h_n^2 r_n^2 + h_{n-1}^2 r_{n-1}^2$ . Canceling  $h_n^2 r_n^2$  from both sides and moving the  $-2h_n^2 r_n$  term to the other side,  $2h_n^2 r_n = h_n^2 + h_{n-1}^2 r_{n-1}^2$  and so, dividing by  $2h_n^2$  on both sides and applying  $1 - h_{n-1}^2 r_{n-1}^2 = h_n^2$ , we have  $r_n = \frac{1 - h_{n-1}^2 r_{n-1}^2 + h_{n-1}^2 r_{n-1}^2}{2h_n^2} = \frac{1}{2h_n^2}$ . We can now substitute this into

our previous formula for  $h_n$ , and we get that  $h_n = \sqrt{1 - h_{n-1}^2 r_{n-1}^2} = \sqrt{1 - h_{n-1}^2 (\frac{1}{2h_{n-1}^2})^2} = \sqrt{1 - \frac{1}{4h_{n-1}^2}}$ . We can define sequence  $s$  such that  $s_x = 1 - \frac{1}{4s_{x-1}}$ , with  $s_1 = 1$  so  $s_x = h_x^2$ . Now, we will perform induction on this to show it is equal to the closed form formula  $\frac{x+1}{2x}$ . For the initial case,  $x = 1$ , we have  $s_1 = \frac{1+1}{2 \cdot 1} = 1$ , which is correct. For any higher  $x$ , we have  $s_{x+1} = \frac{x+1+1}{2(x+1)} = \frac{x+2}{2x+2}$ , and  $1 - \frac{1}{4(\frac{x+1}{2x})} = 1 - \frac{x}{2x+2} = \frac{x+2}{2x+2}$ , so we have successfully proved that  $s_x = \frac{x+1}{2x}$ . Therefore,  $h_x = \sqrt{s_x} = \sqrt{\frac{x+1}{2x}} = \sqrt{\frac{x+1}{x} \cdot \frac{1}{2}}$ .

Therefore, we get that  $\prod_{i=1}^n h_i = \sqrt{\frac{1}{2^n} \frac{n+1}{1}} = \sqrt{\frac{n+1}{2^n}}$ . Now, we plug this into

$$f_n(x) = \frac{x^n}{n!} \prod_{i=1}^n h_i, \text{ and so we have } f_n(x) = \frac{x^n \sqrt{\frac{n+1}{2^n}}}{n!}.$$